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The asymptotic theory of hypersonic boundary-layer stability

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In this paper the linear stability of the hypersonic boundary layer is considered in the local-parallel approximation. It is assumed that the Prandtl-number $\frac{1}{2} < \sigma < 1$ and the viscosity–temperature law is a power function: $\mu/\mu_\infty = (T/T_\infty)^\omega$. The asymptotic theory in the limit $M_\infty \to \infty$ is developed.

Smith & Brown found for the Blasius base flow and Balsa & Goldstein for the mixing layer that, in this limit, the disturbances of the vorticity mode are located in the thin region between the boundary layer and the external flow. The gas model with $\sigma = 1$, $\omega = 1$ was exploited in these studies. Here it is demonstrated that the vorticity mode also exists for gas with $\frac{1}{2} < \sigma < 1$, $\omega < 1$, but its structure and characteristics are considerably different. The nomenclature is discussed, i.e. what an acoustic mode and a vorticity mode are. The numerical solution of the inviscid instability problem for the vorticity mode is obtained for helium and compared with the solution of the complete Rayleigh equation at finite Mach numbers.

The limit $M_\infty \to \infty$ in the local-parallel approximation for the Blasius base flow is considered so as to understand the viscous structure of the vorticity mode. The viscous stability problem for the vorticity mode is formulated under these assumptions. The problem contains only a single similarity parameter which is a function of the Mach and Reynolds numbers, the temperature factor and wave inclination angle. This problem is numerically solved for helium. The universal upper branch of the neutral curve is obtained as a result. The asymptotic results are compared with the numerical solutions of the complete problem.

1. Introduction

Early investigations of compressible boundary-layer stability include those of Lees & Lin (1946) and Lees (1947). An important contribution was made by Mack (1969, 1987), who thoroughly documented inviscid and viscous results of numerical studies. The present state of the problem is discussed by Gaponov & Maslov (1980) and Zhigulev & Tumin (1987).

Recently, there has been much interest in the boundary-layer stability at large values of the Mach number, provoked by recent plans to construct new trans-atmospheric aircraft. The design of this new aircraft differs from previous ones, in that more emphasis has to be placed on the very high speed in the dense layers of atmosphere. For this aircraft the most important problem is the heat transfer and consequently the instability, transition and turbulence in the high-Mach-number boundary layer are involved. The asymptotic studies of these phenomena in the limit $M_\infty \to \infty$ will most likely enable the development of simple practical models suitable for practical use.

Some very significant progress has recently been made in these studies. An asymptotic triple-deck theory for the viscous-mode waves at sufficient angle to the free-stream direction was developed by Smith (1989). The interaction of these waves with the shock was studied by Cowley & Hall (1990). The modification of the triple-deck theory for a strong surface cooling was investigated by Seddougui, Bowles & Smith (1991).

The asymptotic theory of the inviscid-mode instability has also been intensively developed. Cowley & Hall (1990) and Smith & Brown (1990) introduced an asymptotic description of the acoustic modes, and Goldstein & Wundrow (1990) studied the nonlinear problem for these modes. Certain aspects of theoretical research into hypersonic flow stability in boundary layers, shock layers and nozzle flows were described by Brown *et al.* (1991).

We think that the most noteworthy finding was the inviscid vorticity-mode asymptotic description obtained by Smith & Brown (1990) for the Blasius base flow and by Balsa & Goldstein (1990) for the mixing layer. This finding opens a new region for further investigations and may even be employed in practical applications. The gas model with constant heat capacity was used in these papers. The Prandtl number was defined as unity ($\sigma = 1$) and a linear viscosity–temperature law $\mu/\mu_\infty = T/T_\infty$ was chosen.

A very important factor in the asymptotic theory of the stability at $M_\infty \to \infty$ is the choice of the gas model. In this paper the asymptotic theory of hypersonic boundary-layer stability is developed for gas with $\frac{1}{2} < \sigma < 1$, $\mu/\mu_\infty = (T/T_\infty)^\omega$, $\omega < 1$. Helium wind tunnels are the only devices in which very high Mach numbers may be obtained under ideal gas flow. Our gas model is quite suitable for experiments in a helium wind tunnel at large values of Mach numbers. The description of the limit $M_\infty \to \infty$ for this model is proved to be simpler (asymptotic expansions do not contain logarithmic terms). The case with $\omega = 1$ (Balsa & Goldstein; Smith & Brown) is singular for this theory. Therefore, the mathematical statement and physical results are considerably different.

We will further see that the results of the asymptotic theory are very sensitive to the choice of gas model: which part of the asymptotic results for the vorticity mode is stable and which is dependent? To answer this question, we perform the parametric study of the limits for different σ, ω, although most of the numerical results are obtained for helium ($\sigma = \frac{5}{3}$, $\omega = 0.647$). While revising this paper we became familiar with the papers of Blackaby, Cowley & Hall (1990, 1993) where the asymptotic theory for a gas using Sutherland's viscosity law and $\sigma = 1$ was developed. The structure of their base flow is similar to those considered in our paper; therefore, while there is some overlap with our work, there are also many differences. We have studied the viscous counterpart of the inviscid vorticity mode, while Blackaby *et al.* (1990, 1993) have obtained the inviscid vorticity mode in the strong-interaction region. The parametric study at different σ, ω allowed us to discern the importance of the long-wave-limit similarity parameter $s = (1-2\omega)/(1-\sigma(1-\omega)/(1+\omega))$ and to demonstrate that the asymptotic form depends on whether $s < \frac{1}{2}$ or $s > \frac{1}{2}$ (see Grubin & Trigub 1993).

In this paper we also formulate and solve the problem for the viscous vorticity mode in the local-parallel approximation for the non-interactive Blasius base flow. We studied the problem with the aim of demonstrating the existence of the viscous counterpart of the inviscid vorticity mode and to understand its structure. Using the local-parallel approximation most numerical and practical results at high Mach numbers were obtained. Therefore, the asymptotic study of the problem in the local-parallel approximation is of independent significance. Though using the local-parallel approximation, we consider the viscous vorticity-mode analysis as an important building block for hypersonic boundary-layer theory.

The problem formulated in this way contains only a single similarity parameter which is the function of the Mach and Reynolds numbers, the temperature factor and wave inclination angle. The problem is solved numerically for helium. The universal upper branch of the neutral curve and the line of maximum amplification rate are obtained. It is possible to recalculate stability characteristics at various Reynolds and Mach numbers, temperature factors and wave angles. The results are in a good agreement with the numerical results for the complete problem at $M_\infty = 20$.

The plan of the paper is as follows. The limit $M_\infty \to \infty$ for the boundary layer on a flat plate far enough from the leading edge to eliminate all interaction effects is considered in §2. The asymptotic expansions of the velocity and temperature profiles in the boundary and transition layers are obtained. The transition layer is the thin region between the boundary layer and the external flow. It is shown that the generalized inflexion point is situated in the transition layer if $\sigma > \frac{1}{2}$. The profiles in the transition layer were numerically obtained for helium. The influence of the temperature factor on the profiles is investigated and discussed.

The compressible boundary-layer linear stability problem in the local-parallel approximation is briefly described in §3, using the operator form of the equations. This part does not contain new results but provides important background and introduces most of the notation.

The inviscid instability problem for the vorticity mode at $M_\infty \to \infty$ is formulated in §4. The definition of the vorticity mode is considered in detail. The phase velocity and amplification rate as functions of the wavenumber for the vorticity mode are numerically obtained for helium. A comparison between the results for the neutral inflexional modes at finite M_∞ and for the asymptotic vorticity mode is also presented.

The viscous instability problem for the vorticity mode at $M_\infty \to \infty$ in the local-parallel approximation is formulated in §5. The problem is solved numerically with the aid of the spectral method. The universal upper branch of the neutral curve is obtained and compared with the numerical results for the complete problem. The main features of the vorticity mode in the limit $M_\infty \to \infty$ are assigned and discussed.

2. The velocity and temperature profiles in the non-interactive boundary layer on a flat plate as $M_\infty \to \infty$

Consider the origin of rectangular coordinates (x_d, y_d, z_d) at the leading edge of the plate, where the axis is directed along the plate, y_d is normal to plate and z_d is the transverse axis along the edge. The subscript d denotes dimensional variables and ∞ the free-stream values. The plate is in a flow of gas with constant Prandtl number σ and $\gamma = C_p/C_v$. The viscosity–temperature law is $\mu_d/\mu_\infty = (T_d/T_\infty)^\omega$, where $\omega < 1$. The lengthscale $L_d = x_d/R$, where Reynolds number $R = (u_\infty \rho_\infty x_d/\mu_\infty)^{\frac{1}{2}}$, is of the same order as the boundary-layer thickness for $M_\infty = O(1)$.

We study the flow near the plate very far from the leading edge where interaction effects do not influence the leading-order approximation (the correlation between R and M_∞ in this region will be considered further below). The base flow there is described by the similarity solution of the boundary-layer equations (Hayes & Probstein 1959):

$$\left.\begin{aligned} &(Nf'')'+ff''=0,\\ &(1/\sigma)(NT')'+fT'+(2/\epsilon)Nf''^2=0,\\ 0<\eta<\infty,\quad &f(0)=f'(0)=0,\quad f'(\infty)=1,\quad T(0)=T_w,\quad T(\infty)=1,\end{aligned}\right\}\qquad(2.1)$$

where
$$\eta=\frac{1}{\sqrt{2}}\int_0^y\frac{\mathrm{d}y}{T},\quad y_d=L_d y,\quad u_d=u_\infty f'(\eta),$$

$$T_d=T_\infty T(\eta),\quad \epsilon=\frac{2}{(\gamma-1)M_\infty^2},\quad N=T^{-(1-\omega)};$$

u_d is the streamwise component of velocity, and primes denote differentiation.

Freeman & Lam (1959) showed that the non-interactive boundary layer divides into two layers as $\epsilon\to 0$. In the inner (boundary) layer (subscript b) of thickness $y\sim O(\epsilon^{-(1+\omega)/2})$ gas is strongly heated and velocity varies from zero to its value in the free stream:

$$\left.\begin{aligned} T&=\frac{1}{\epsilon}(T_b(\eta_b)+O(\epsilon^{\bar\beta/\lambda_2})),\\ f&=\epsilon^{1/\lambda_2}(f_b(\eta_b)+O(\epsilon^{\bar\beta/\lambda_2})),\\ &(N_b f_b'')'+f_b f_b''=0,\\ &\frac{1}{\sigma}(N_b T_b')'+f_b T_b'+2N_b f_b''^2=0,\end{aligned}\right\}\qquad(2.2)$$

where

$$0<\eta<\infty,\quad f_b(0)=f_b'(0)=0,\quad f_b'(\infty)=1,\quad T_b(0)=T_{bw},\quad T_b(\infty)=0,$$

$$\eta_b=\frac{\eta}{\epsilon^{1/\lambda_2}},\quad N_b=T_b^{-(1-\omega)},\quad \bar\beta=\frac{(2-\omega^2)^{\frac{1}{2}}+\omega}{1-\omega},$$

$$y=\epsilon^{-(1+\omega)/2}\sqrt{2}\int_0^{\eta_b}T_b\,\mathrm{d}\eta_b=\epsilon^{-(1+\omega)/2}y_b,\quad \lambda_2=\frac{2}{1-\omega}.$$

The boundary point $\eta_b=\infty$ is singular. The asymptotic expansions of functions in its neighbourhood are

$$\left.\begin{aligned} &f_b=\xi+C_1\xi^{-\lambda_1}(1+O(\xi^\beta)),\qquad T_b=C_2\xi^{-\lambda_2}(1+C_3\xi^\beta+O(\xi^{2\beta})),\\ &\xi=\eta_b-C_0,\quad \lambda_1=\frac{1}{\sigma}\frac{1+\omega}{1-\omega},\qquad \beta=-\frac{(2-\omega^2)^{\frac{1}{2}}-\omega}{1-\omega},\quad C_2=\lambda_1^{1/(1-\omega)}.\end{aligned}\right\}\qquad(2.3)$$

There are three arbitrary constants, C_0, C_1, C_3, which can be determined from matching (2.3) and the solution of (2.2). All subsequent terms in the expansions (2.3) include only these arbitrary constants.

The large number of terms in (2.3) have been used in a numerical analysis of perturbations in the inner layer (Grubin & Trigub 1993). The subsequent structure of series (2.3) and the method of calculation are described in Appendix A.

It has proved helpful to use the Dorodnitsyn–Howart variable η in the stability calculations. However, a more clear representation is given with the coordinate y as an independent variable. We will use both these representations and show the connection between them. An interesting feature of the solution of (2.2) is that the edge of the layer ($\eta_b \to \infty$) is situated at a finite value of ordinate y_b

$$y_{b\infty} = \surd 2 \int_0^\infty T_b \, d\eta_b.$$

As $\eta_b \to \infty$

$$y \to \epsilon^{-(1+\omega)/2} y_{b\infty},$$

$$\xi = \bar{y}^{-1/(\lambda_2-1)} \left(\frac{\surd 2 C_2}{\lambda_2 - 1}\right)^{1/(\lambda_2-1)} (1 + O(\bar{y}^{-\beta/(\lambda_2-1)})), \tag{2.4}$$

$$\bar{y} = \epsilon^{-(1+\omega)/2} y_{b\infty} - y.$$

It is the solution of (2.2) that is important for most practical applications, i.e. when only friction, heat transfer or other integral characteristics of a boundary layer are required, it gives the main approximation. However (as shown later), in the hypersonic stability theory, profiles in the outer transition layer (subscript t) have a major role. This thin layer ($\delta_t/\delta_b = O(\epsilon^{(1+\omega)/2})$) is intermediate between the inner boundary layer and the free stream. Here there is a weak deceleration and strong heating of the outer flow due to viscous forces. Bush (1966) demonstrated that an analogous transition layer also exists in the strong-interaction regime ($\chi = M_\infty^{2+\omega}/R \to \infty$). The difference with our study is that Bush's solution was matched with the entropy layer and not with the free stream. The transition layer also exists in the weak-interaction regime $\chi = o(1)$ (Bush & Cross 1967). Thus, the transition layer originates just from the beginning of the strong-interaction region. Therefore it may be possible to transfer the results of the stability theory for the transition layer to the moderate- and strong-interaction regimes (as done by Blackaby *et al.* 1990, 1993).

In the transition layer $\eta_t \equiv \eta - C_0 \epsilon^{(1-\omega)/2} = O(1)$,

$$f = \eta_t + \epsilon^\nu [f_t(\eta_t) + O(\epsilon^{-\beta/\lambda_2})],$$

$$T = T_t(\eta_t) + O(\epsilon^{-\beta/\lambda_2}), \quad \nu = (1+\omega)/2\sigma + \tfrac{1}{2}(1-\omega),$$

$$0 < \eta_t < \infty.$$

The temperature equation may be considered separately:

$$\frac{1}{\sigma}(T_t'/T_t^{1-\omega})' + \eta_t T_t' = 0. \tag{2.5}$$

The asymptotic expansion of the solution of (2.5) as $\eta_t \to 0$ is

$$T_t = C_2 \eta_t^{-\lambda_2}(1 + C_4 \eta_t^{\bar{\beta}} + \ldots), \tag{2.6}$$

where C_4 is an arbitrary constant which must be chosen so that the condition $T_t(\infty) = 1$ is valid.

It is easy to obtain the numerical solution of (2.5) since the boundary-value problem for T_t allows the group transformation. Denote as $\tilde{T}_t(\eta_t)$ the solution of the Cauchy problem for (2.5) with the initial values from (2.6) and $C_4 = \tilde{C}_4$. Then if $\tilde{T}(\eta_t) \to \tilde{T}_\infty$ as $\eta_t \to \infty$ the solution of the boundary-value problem is

$$T_t(\eta_t) = \frac{1}{\tilde{T}_\infty} \tilde{T}_t\left(\frac{\eta_t}{\tilde{T}_\infty^{(1-\omega)/2}}\right), \quad C_4 = \tilde{C}_4(\tilde{T}_\infty)^r,$$

$$r = -\tfrac{1}{2}[(2-\omega^2)^{\frac{1}{2}} + \omega].$$

The function f_t' satisfies the linear equation

$$(f_t''/T_t^{1-\omega})' + \eta_t f_t'' = 0, \tag{2.7}$$

$$f_t'(\infty) = 0, \quad f_t'' = \lambda_1(\lambda_1+1)\,C_1\,\eta_t^{-\lambda_1-2}\ \ldots \quad \text{as} \quad \eta_t \to 0.$$

When the profile of T_t is known the profile of the velocity defect is

$$f_t'(\eta_t) = -C_1\,u_t(\eta_t)$$

$$u_t = (\lambda_1+1)\int_{\eta_t}^{\infty} T_t^{1-\omega}(\eta)\,\mathrm{e}^{-\eta^2/2}\left(\int_{\eta}^{\infty} t(T_t^{1-\omega}(t)-1)\,\mathrm{d}t\right)\mathrm{d}\eta.$$

The relation between the coordinates y and η_t in the transition layer is

$$y = \epsilon^{-(1+\omega)/2} y_{b\infty} + y_t + o(1), \tag{2.8}$$

$$y_t = -\frac{\sqrt{2}C_2\,\eta_t^{-(\lambda_2-1)}}{\lambda_2-1} + \sqrt{2}\eta_t - \sqrt{2}\int_0^{\eta_t}(1-T_t(\eta)+C_2\,\eta^{-\lambda_2})\,\mathrm{d}\eta.$$

Thus if $\eta_t \to 0$, then $y_t \to -\infty$,

$$\left.\begin{aligned}
\eta_t &= (-y_t)^{-1/(\lambda_2-1)}\left(\frac{\sqrt{2}C_2}{\lambda_2-1}\right)^{1/(\lambda_2-1)}(1+O((-y_t)^{-\beta/(\lambda_2-1)})),\\
T_t &= \bar{C}_2(-y_t)^{k_2}(1+O((-y_t)^{-\beta/(\lambda_2-1)})),\\
u_t &= \bar{C}_1(-y_t)^{k_1}(1+O((-y_t)^{-\beta/(\lambda_2-1)})),
\end{aligned}\right\} \tag{2.9}$$

where

$$k_1 = \frac{1}{\sigma}+\frac{1-\omega}{1+\omega}, \quad k_2 = \frac{2}{1+\omega}, \quad \bar{C}_1 = \lambda_1\left(\frac{\sqrt{2}C_2}{\lambda_2-1}\right)^{-k_1}, \quad \bar{C}_2 = C_2\left(\frac{\sqrt{2}C_2}{\lambda_2-1}\right)^{-k_2}.$$

Now we state conditions for when the generalized inflexion point is in the transition layer. According to the stability theory this point defines the position of the critical layer for the inflexional neutral modes family (Mack 1969). The following equation determines the point:

$$F_s = f'''(\eta) - 2\frac{T'(\eta)}{T(\eta)}f''(\eta) = 0. \tag{2.10}$$

Taking into account (2.7), (2.10) is rewritten in the layer $\eta_t = O(1)$:

$$F_s = -\epsilon^{\nu}\frac{f_t''(\eta_t)}{\eta_t}\bar{F}_s(\eta_t), \quad \bar{F}_s = \eta_t^2\,T_t^{1-\omega} + (1+\omega)\frac{T_t'}{T_t}\eta_t.$$

The function $f_t'' > 0$ at $0 < \eta_t < \infty$; $\bar{F}_s = \eta_t^2 + O(1)$ as $\eta_t \to \infty$, and

$$\bar{F}_s = \frac{1+\omega}{1-\omega}\left(\frac{1}{\sigma}-2\right) + O(1) \quad \text{as} \quad \eta_t \to 0.$$

Consequently if $0 < \omega < 1$, $\sigma > \frac{1}{2}$, the function $\bar{F}_s(\eta_t)$ changes sign at $\eta_t = O(1)$.

All the calculations presented in this paper have been performed for helium ($\gamma = \frac{5}{3}$,

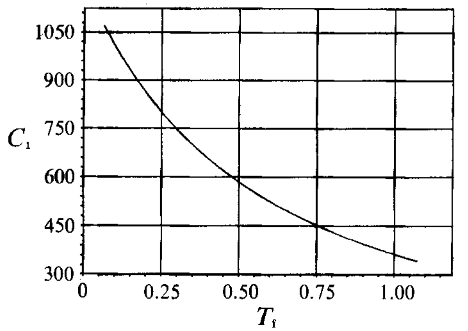

FIGURE 1. The dependence of the coefficient C_1 on the temperature factor $T_f = T_w/T_r$. C_1 determines the intensity of the velocity defect in the transition layer: $U(y) = 1-\epsilon^{\nu} C_1 u_t(y_t)$.

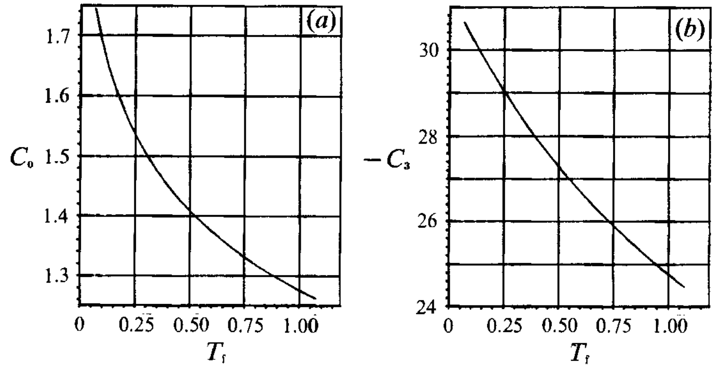

FIGURE 2. The coefficients C_0, and $-C_3$ in the expansions (2.3) as functions of the temperature factor.

$\sigma = \frac{2}{3}$, $\omega = 0.647$). Figure 1 presents the relation between C_1 and the temperature factor $T_f = T_w/T_r$, where T_r is the adiabatic wall temperature. The value of C_1 defines the intensity of the velocity defect in the transition layer. This is the only information from the boundary layer entering the leading-order approximation for the transition layer. The wall cooling leads to a sharp growth of the velocity defect near the generalized inflexion point, C_1 rapidly increases, and perhaps tends to infinity (an asymptotic analysis of the limit $T_f \to 0$ was not carried out). Such behaviour is inconsistent with the assumptions made in the problem involved. Therefore the limit of low temperature factor requires special investigation. Figure 2 shows C_0 and $-C_3$ as functions of T_f. These values enter the higher-order terms of the profile expansions in the transition layer. Their increase at low T_f can cause a large contribution from higher terms and destroy the first-order approximation.

In figure 3 the profiles of temperature $T_t(y_t)$, velocity $u_t(y_t)$ and their second derivatives $T_t''(y_t)$, $u_t''(y_t)$ are shown. The dashed line denotes the generalized inflexion point position $y_{ts} = 0.684$. The temperature at this point for helium $T_t(y_{ts}) \approx 4.5$. The second derivatives have maxima in the transition layer and tend to zero as $y_t \to \pm\infty$. The transition layer considered with the boundary-layer thickness scale resulted in a discontinuity of the second derivatives. As $y_b \to y_{b\infty}$, $T_b'' \to +\infty$ and $f_b''' \to \infty$; and for $y_b > y_{b\infty}$, $T_b''' = 0$ and $f_b''' = 0$.

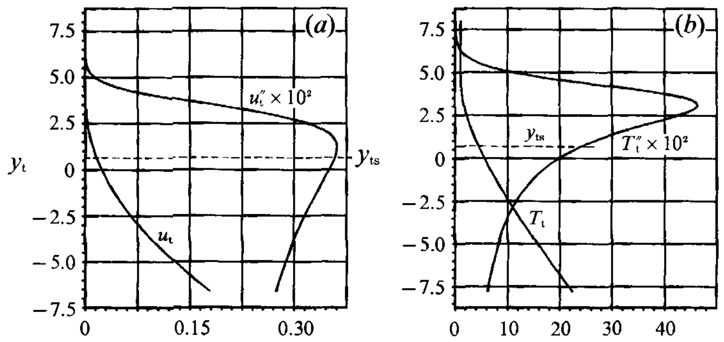

FIGURE 3. The universal profiles of the temperature $T_t(y_t)$ and the velocity defect $u_t(y_t)$ and their second derivatives in the transition layer. The second derivatives clearly demonstrate that the transition layer has its own lengthscale. The dashed line shows the position of the generalized inflexion point $y_{ts} = 0.684$, $T_t(y_{ts}) = 4.5$

3. The linear stability problem theory in the local-parallel approximation

Most of the numerical results on boundary-layer stability at moderate and high Mach numbers were obtained for a non-interactive boundary layer in the local-parallel approximation. Here the most common notation, as stated by Mack (1969), is used. Velocity, temperature, density, pressure and viscosity coefficients are made dimensionless by their values in the free stream, and length by L_d. In the local-parallel approximation we have a system of linear partial differential equations for small perturbations of the streamwise, normal and spanwise components of velocity (u', v', w'), pressure p', density ρ' and temperature T'. These equations contain the local profiles of streamwise $U(y)$ and spanwise $W(y)$ velocity and temperature $T(y)$. The complex amplitude functions of the disturbances are defined by

$$(u', v', w', p', \rho', T') = (f, \alpha\phi, h, \pi, r, \theta)\exp[\mathrm{i}(\alpha x + \beta z - \alpha c t)].$$

The following transformation is used:

$$\begin{aligned}
&\psi = \arctan \beta/\alpha,\\
&\tilde{U} = U + W\tan\psi, \quad \tilde{W} = W - U\tan\psi,\\
&\tilde{x} = x\cos\psi + z\sin\psi, \quad \tilde{z} = -x\sin\psi + z\cos\psi,\\
&\tilde{f} = f + h\tan\psi, \quad \tilde{h} = h - f\tan\psi, \quad \tilde{\phi} = \phi, \quad \tilde{\pi} = \pi,\\
&\tilde{r} = r, \quad \tilde{\theta} = \theta, \quad \tilde{M} = M_\infty\cos\psi, \quad \tilde{R} = R\cos\psi,\\
&\tilde{\alpha} = \alpha/\cos\psi, \quad \tilde{c} = c, \quad \tilde{t} = t\cos\psi, \quad \alpha x + \beta z - \alpha c t = \tilde{\alpha}\tilde{x} + \tilde{\alpha}\tilde{c}\tilde{t}.
\end{aligned}$$

Thus we have a system of ordinary differential equations (see Mack 1969), which may be reduced to a compact form, simplifying its analysis.

Denote

$$\tilde{d} \equiv \mathrm{i}\tilde{f} + \tilde{\phi}', \quad \tilde{P} \equiv \frac{\tilde{\pi}}{\gamma\tilde{M}^2} + \tfrac{2}{3}(\mu - \lambda)\frac{\tilde{\alpha}}{\tilde{R}}\tilde{d},$$

$$\tilde{D} \equiv \frac{(\gamma-1)\tilde{M}^2}{\tilde{\alpha}\tilde{R}}[2\mu\tilde{U}'(\tilde{f}' + \mathrm{i}\tilde{\alpha}^2\tilde{\phi}) + \tilde{s}(\tilde{U}'^2 + \tilde{W}'^2) + 2\mu\tilde{W}'\tilde{h}'].$$

We define new differential operators which act on the left-hand sides of the functions:

$$\Delta^-(g) \equiv \frac{\mathrm{d}}{\mathrm{d}y} g \frac{\mathrm{d}}{\mathrm{d}y} - \tilde{\alpha}^2 g, \quad \Delta^+(g) \equiv \frac{\mathrm{d}}{\mathrm{d}y} g \frac{\mathrm{d}}{\mathrm{d}y} + \tilde{\alpha}^2 g,$$

$$L_\theta = \frac{\tilde{U}-\tilde{c}}{T} - \frac{1}{\mathrm{i}\sigma\tilde{\alpha}\tilde{R}} \Delta^-(1)\,\mu, \quad L_\phi = \frac{\tilde{U}-\tilde{c}}{T} - \frac{1}{\mathrm{i}\tilde{\alpha}\tilde{R}} \Delta^-(\mu).$$

Using these definitions we rewrite equations in the form

$$\left.\begin{aligned}
L_\phi \tilde{f} &= -\tilde{P} + \mathrm{i}\frac{\tilde{U}'}{T}\tilde{\phi} - \frac{1}{\tilde{\alpha}\tilde{R}}[\mathrm{i}(\tilde{s}\tilde{U}')' - \tilde{\alpha}^2(\mu\tilde{d} + \mu'\tilde{\phi})],\\
L_\phi \tilde{\phi} &= \mathrm{i}\frac{\tilde{P}'}{\tilde{\alpha}^2} + \frac{1}{\tilde{\alpha}\tilde{R}}[\tilde{s}\tilde{U}' - \mathrm{i}(\mu\tilde{d}' + \mu'\tilde{\phi}')],\\
L_\phi \tilde{h} &= \mathrm{i}\frac{\tilde{W}'}{T}\tilde{\phi} - \frac{\mathrm{i}}{\tilde{\alpha}\tilde{R}}(\tilde{s}\tilde{W}')',\\
L_\theta \tilde{\theta} &= \frac{\gamma-1}{\gamma}(\tilde{U}-\tilde{c})\,\tilde{\pi} + \mathrm{i}\frac{T'}{T}\tilde{\phi} + \tilde{D},\\
\frac{1}{\gamma}(\tilde{U}-\tilde{c})\,\tilde{\pi} - \mathrm{i}\tilde{d} &= \frac{1}{\mathrm{i}\sigma\tilde{\alpha}\tilde{R}}\Delta^-(1)\,\mu\tilde{\theta} + \tilde{D}.
\end{aligned}\right\} \quad (3.1)$$

Here λ is the bulk viscosity, $\tilde{s} = \tilde{\theta}\,\mathrm{d}\mu/\mathrm{d}T$, and primes denote differentiation. This form also proved to be convenient for calculations using spectral methods.

To consider subsonic or outgoing and amplified supersonic waves we should use decay conditions as $y \to \infty$ (Mack 1969):

$$\tilde{f}, \tilde{\phi}, \tilde{h}, \tilde{\pi}, \tilde{\theta} \to 0. \quad (3.2)$$

At $y = 0$ the boundary conditions are

$$\tilde{f} = \tilde{\phi} = \tilde{h} = \tilde{\theta} = 0. \quad (3.3)$$

For the temporal problem $\tilde{\alpha}$ and ψ are parameters. Eigenvalues $\tilde{c}$ and corresponding eigenfunctions which satisfying the system of equations (3.1) and the boundary conditions (3.2), (3.3) are to be obtained.

In the limit $\tilde{R} \to \infty$, equations of inviscid stability theory will be obtained. The problem can be formulated as

$$\left.\begin{aligned}
\tilde{\pi}'' + \left(\frac{T'}{T} - 2\frac{\tilde{U}''}{\tilde{U}-\tilde{c}}\right)\tilde{\pi}' - \tilde{\alpha}^2\left(\frac{T - \tilde{M}^2(\tilde{U}-\tilde{c})^2}{T}\right)\tilde{\pi} &= 0,\\
\pi'(0) = \pi(\infty) &= 0.
\end{aligned}\right\} \quad (3.4)$$

The other functions can be expressed using $\tilde{\pi}$:

$$\left.\begin{aligned}
\tilde{\phi} &= \mathrm{i}\frac{\tilde{\pi}' T}{\tilde{\alpha}^2\gamma\tilde{M}^2(\tilde{U}-\tilde{c})}, \quad \tilde{f} = -\frac{\tilde{\pi}}{\gamma\tilde{M}^2}\frac{T}{\tilde{U}-\tilde{c}} + i\frac{\tilde{U}'}{\tilde{U}-\tilde{c}}\tilde{\phi},\\
\tilde{\theta} &= \frac{\gamma-1}{\gamma}T\tilde{\pi} + \mathrm{i}\frac{T'}{\tilde{U}-\tilde{c}}\tilde{\phi}, \quad \tilde{h} = \mathrm{i}\frac{\tilde{W}'\tilde{\phi}}{\tilde{U}-\tilde{c}}.
\end{aligned}\right\} \quad (3.5)$$

The profiles for an undisturbed flow in the boundary layer are

$$\left.\begin{aligned} &\tilde{U} = f_b'(\eta_b) \equiv 1 - u_b(\eta_b) \\ &\tilde{T} = \frac{1}{\epsilon} T_b(\eta_b), \quad \frac{d}{dy} = \epsilon^{(1+\omega)/2} \frac{1}{\sqrt{2T_b}} \frac{d}{d\eta_b}; \end{aligned}\right\} \tag{3.6}$$

and in the transition layer

$$\left.\begin{aligned} &\tilde{U} = 1 + \epsilon^{\nu} f_t'(\eta_t) \equiv 1 - \epsilon^{\nu} C_1 u_t(\eta_t), \\ &\tilde{T} = T_t(\eta_t), \quad \frac{d}{dy} = \frac{d}{dy_t} = \frac{1}{\sqrt{2T_t}} \frac{d}{d\eta_t}. \end{aligned}\right\} \tag{3.7}$$

4. Vorticity mode; inviscid analysis

It was Mack (1969) who noticed the irregular parts in the $\tilde{\alpha}(M_\infty)$ dependence for the inflexional neutral modes. While the general tendency was a decrease of $\tilde{\alpha}$ when M_∞ increases, the non-typical parts showed the inverse (two of the non-typical parts exist for the first mode and one for every subsequent mode). Mack noticed that at large values of Mach number the end of the non-typical part for the mode number n approaches the beginning of the non-typical part for the mode number $n+1$. Thus, the non-typical parts form a continuous curve at $M \gg 1$ almost everywhere. The important observation was made that an eigenfunction for $(\tilde{\alpha}, M_\infty)$ corresponding to the non-typical part differed from one belonging to the typical $(\tilde{\alpha}, M_\infty)$ part. An eigenfunction $\tilde{\pi}(y)$ from the non-typical part has its largest extremum in the generalized inflexion point and small amplitude in the inner part of the boundary layer. For all other $(\tilde{\alpha}, M_\infty)$ values the eigenfunction has the largest extremum in the inner layer, and rapidly decays near the outer edge of the boundary layer. All these facts indicate that a special mode exists at $M_\infty \gg 1$.

An asymptotic theory for $\sigma = 1$, $\omega = 1$ was constructed by Smith & Brown (1990). The theory describes the near-linking of the non-typical parts at $M_\infty \to \infty$ and the formation of a continuous neutral line almost everywhere. The mode corresponding to this line was called the 'vorticity mode'. Lees (1947) used this term for the inflexional mode at small values of M_∞ when there are no other modes. The nature of the neutral vorticity mode is the same as for the neutral mode in an incompressible fluid with the presence of an inflexional point. Therefore, we believe that 'hydrodynamic mode' would be a more appropriate designation. Smith & Brown (1990) showed that the perturbations of the vorticity mode are localized in the neighbourhood of the outer edge of boundary layer, the transition layer in our paper.

In order to apply the term 'vorticity mode' to the whole range of the M_∞ values, Smith & Brown proposed that all the parts of $\tilde{\alpha}(M_\infty)$ dependence where $d\tilde{\alpha}/dM_\infty > 0$ belong to the vorticity mode. However, it is shown below that there are values of σ and ω for which this definition is not satisfactory. Since the vorticity mode is of importance in hypersonic stability theory, we will give a new definition which is more suitable to the essence of the phenomenon.

We can assume that the function $\tilde{\pi}(y)$ is real for the neutral mode, $\tilde{c} = \tilde{c}_s \equiv \tilde{U}(y_s)$, where y_s is the coordinate of the generalized inflexion point. The function $|\tilde{\pi}(y)|$ has at least one maximum in the interval $0 \leqslant y < \infty$. The conditions for a maximum in a regular point are $\tilde{\pi}' = 0$ and $\tilde{\pi}'' < 0$. Also, a maximum in the singular point is possible. Therefore, we may conclude from equation (3.4) that only two kinds of maxima can exist:

(i) the maxima in the region where

$$\tilde{M}^2(\tilde{U}-\tilde{c}_s)/T > 1;$$

(ii) the maximum at the generalized inflexion point where the general form of solution of (3.4) is

$$\tilde{\pi}(y) = a(1-\tfrac{1}{2}\tilde{\alpha}^2x^2+O(x^4))+b(x^3+O(x^4));$$

where $x = y-y_s$; and a, b are arbitrary constants.

Maxima of kind (i) are related to acoustic oscillations in the region where the wave speed is supersonic with respect to the undisturbed flow. These oscillations are impossible at small values of M_∞ ($M_\infty \leqslant 2$ in Mack 1969). The maximum of kind (ii) is connected with the vorticity motion and can exist at any value of M_∞ in the presence of the generalized inflexion point.

The neutral mode for which the largest maximum of the function $|\tilde{\pi}(y)|$ is in the critical layer at $y = y_s$ will be called the neutral vorticity mode. In the other case, the mode will be called the neutral acoustic mode.

This definition leads to a natural generalization for amplified and damped modes. Let us assume that the function $|\tilde{\pi}|$ has its largest maximum at a certain point. If $|\tilde{\pi}|' = 0$ at this point we then obtain from (3.4) that

$$|\tilde{\pi}|^{2''} = 2(F_1+F_2)\,|\tilde{\pi}|^2,$$

$$F_1 \equiv \tilde{\alpha}^2\left[1-\frac{\tilde{M}^2}{T}((\tilde{U}-\tilde{c}_r)^2-\tilde{c}_i^2)\right], \quad F_2 \equiv 2\frac{\tilde{U}'\tilde{c}_i}{|\tilde{U}-\tilde{c}|}\left(\frac{\tilde{\pi}_r'}{\tilde{\pi}_i}\right).$$

At the maximum, the value $\tilde{\pi}_r'/\tilde{\pi}_i = -\tilde{\pi}_i'/\tilde{\pi}_r$ does not depend on the normalization of $\tilde{\pi}$. The sum $F_1+F_2 < 0$ has the function of 'returning force' which defines the maximum. The function F_1 is related to the acoustics and F_2 is related to the vorticity. The kind of mode is defined depending on the contribution of these two factors to the 'returning force': if $F_1 > 0$ at the point of the largest maximum of $|\tilde{\pi}|$ the mode will be called a vorticity mode; if $F_2 > 0$ it will be called an acoustic mode. When $F_1 < 0$ and $F_2 < 0$, the mode will be called a vorticity mode if $|F_2| > |F_1|$, and an acoustic mode otherwise.

In this way, the definitions introduced reflect the physical mechanism which causes the most intensive pressure pulsations. This definition should be in a good agreement with the essence of the phenomenon in distinctive cases; however, it could be too arbitrary in the marginal ones.

We note an important consequence. If $\tilde{c}_i \ll 1$ and the largest maximum of $|\tilde{\pi}|$ is in the subsonic region ($\tilde{M}^2(\tilde{U}-\tilde{c}_r)^2/T < 1$) then the mode is a vorticity mode.

To investigate the asymptotic form of the inviscid vorticity mode as $\epsilon \to 0$, the local-parallel approximation and non-interactive boundary-layer base profiles are quite justified, because for every large M_∞ we can choose R so large that all non-parallel and interaction effects are negligible. In the transition layer $\tilde{\pi} = \pi_t(\eta_t)+\ldots$, $\tilde{c} = 1-\epsilon^\nu C_1 c_t$, $c_t = O(1)$. If we substitute the transition layer profiles (3.7) into (3.4) and take the limit $\epsilon \to 0$, $(\tilde{\alpha}, c_t, \psi) = O(1)$, we have

$$\pi_t''-2\frac{u_t'}{u_t-c_t}\pi_t'-2\tilde{\alpha}^2T_t^2\pi_t = 0. \tag{4.1}$$

The supposition in this limit is naturally the following:

$$\frac{2\cos^2\psi\, C_1^2}{\gamma-1}\epsilon^{2\nu-1}(u_t-c_t)^2 \ll T_t. \tag{4.2}$$

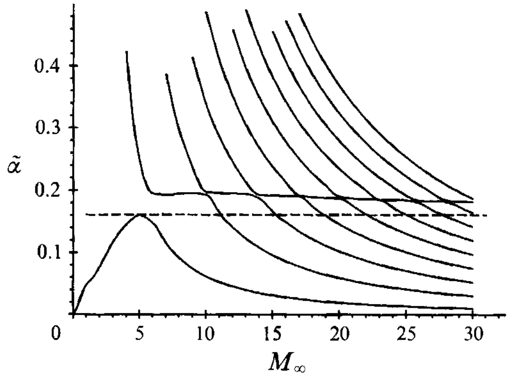

FIGURE 4. The neutral inflexional-mode curves at finite Mach numbers, $T_f = 1$. The asymptotic neutral vorticity-mode wavenumber $\tilde{\alpha} = 0.1603$ is shown as a dashed line.

The power $2\nu - 1 = 1/\sigma + \omega((1/\sigma) - 1) > 0$, and the condition (4.2) is always correct at the limit $\epsilon \to 0$. We note that in the neighbourhood of the critical layer it should be correct even at moderate values of M_∞.

The two linearly independent solutions of (4.1) as $\eta_t \to +\infty$, $y_t \to +\infty$ have the form

$$\pi_{t(1,2)} = \exp(\pm\sqrt{2\tilde{\alpha}\eta_t}) + \ldots = \exp(\pm\tilde{\alpha}y_t) + \ldots .$$

At $\eta_t \to 0, y_t \to -\infty$ the asymptotic expansions of the two linearly independent solutions of (4.1) are

$$\pi_{t(1,2)} = -\frac{u_t}{T_t^{\frac{1}{2}}}\exp\left(\pm\sqrt{2\tilde{\alpha}}\frac{C_2}{\lambda_2 - 1}\eta_t^{-(\lambda_2 - 1)}\right) + \ldots$$
$$= -\frac{u_t}{T_t^{\frac{1}{2}}}\exp(\pm\tilde{\alpha}y_t) + \ldots . \tag{4.3}$$

The method of obtaining (4.3) is given in Appendix B.

We discard the growing solution and obtain the boundary conditions for (4.1):

$$\pi_t \to 0 \quad \text{as} \quad \eta_t \to +\infty; \quad \pi_t \to 0 \quad \text{as} \quad \eta_t \to 0. \tag{4.4}$$

It is necessary to find for any $\tilde{\alpha}$ the values of c_t for which there is a non-zero solution of (4.1) with the uniform boundary conditions (4.4). If the solution is found, we can use (3.5) to obtain expressions for the other functions. Using the variable y_t the main terms of their expansions in the transition layer are given as

$$\left.\begin{array}{c}\tilde{\phi} = Q(\phi_t(y_t) + \ldots), \quad \tilde{f} = Q(f_t(y_t) + \ldots), \\ \tilde{\theta} = (Q/\epsilon^\nu C_1)(\theta_t(y_t) + \ldots), \quad \tilde{h} = \tan\psi Q(h_t(y_t) + \ldots), \\ Q = \dfrac{\gamma - 1}{2\gamma\cos^2\psi C_1\epsilon^{\nu-1}}, \quad \phi_t = \mathrm{i}\dfrac{T_t}{\tilde{\alpha}^2(c_t - u_t)}\pi_t', \\ f_t = -\dfrac{T_t\pi_t + \mathrm{i}u_t'\phi_t}{c_t - u_t}, \quad \theta_t = \mathrm{i}\dfrac{T'}{c_t - u_t}\phi_t, \quad h_t = \mathrm{i}\dfrac{u_t'}{c_t - u_t}.\end{array}\right\} \tag{4.5}$$

It is clear that functions $\varphi_t, f_t, \theta_t, h_t$ also decay exponentially as $y_t \to \pm\infty$.

Figure 4 presents the $\tilde{\alpha}(M_\infty)$ curves for the inflexional modes ($\tilde{c}_r = \tilde{U}(y_s)$), obtained from the numerical solution of (3.4) for helium with $T_f = 1$. The main difference from the cases investigation by Mack and Smith & Brown is that $\mathrm{d}\tilde{\alpha}/\mathrm{d}M_\infty < 0$ on the non-

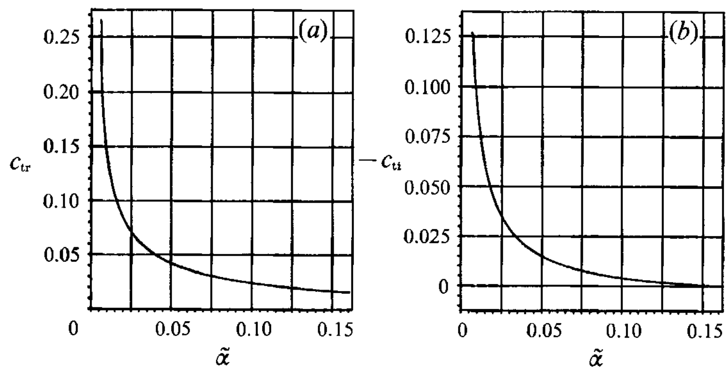

FIGURE 5. The real and imaginary parts of the wave speed as a function of $\tilde{\alpha}$ for the solution of (4.1).

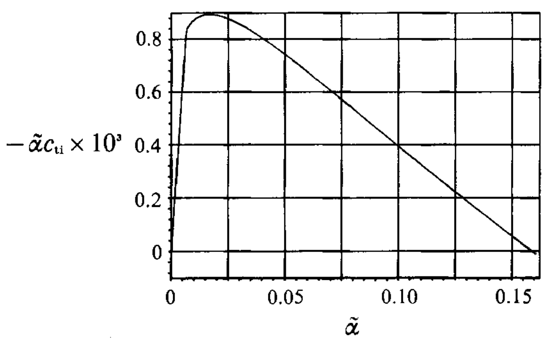

FIGURE 6. The growth rate $-\tilde{\alpha}c_{ti}$ as a function of $\tilde{\alpha}$.

typical parts. The edges of the non-typical parts rapidly draw together and form a straight line $\tilde{\alpha}$ = const. rather than a growing curve. The dashed line indicates the value $\tilde{\alpha} = 0.1603$ obtained from the numerical solution of (4.1) with boundary conditions (4.4) for the neutral disturbances with $c_{tr} = u_t(y_{ts})$ – the neutral vorticity mode at $M_\infty \to \infty$. We think that the main cause of the poor agreement at moderately high Mach numbers is the unsatisfactory precision of the asymptotic profiles in the transition layer. The main approximation for the profiles only has a high precision at high Mach numbers, especially for the temperature profiles.

Numerical investigations of (4.1) for an amplified mode were also carried out. The functions $c_{tr}(\tilde{\alpha})$, $-c_{ti}(\tilde{\alpha})$ are presented in figure 5. Infinite growth is observed as $\tilde{\alpha} \to 0: c_{tr} \sim c_{ti} \sim \tilde{\alpha}^{-k_1/(k_2+1)}$, and the critical layer moves down in the region where large values of temperature and velocity defect occur. This behaviour may be called a 'long-wave catastrophe'. A full investigation of the stability requires the analysis of (3.4) at the limit $\tilde{\alpha} \to 0$. This limit contains new effects: the influence of the perturbations in the boundary layer at $\tilde{\alpha} \sim \epsilon^{(1+\omega)/2}$ and compressibility effects in the outer flow at $c_{tr} \sim 1/\epsilon^{\nu-1/2}$. An analysis of the long-wave limit is presented in a companion paper, Grubin & Trigub (1993).

However, it should be noted that the maximum increment $\max(-C_1 \epsilon^{\nu} c_{ti} \tilde{\alpha})$ is observed only for the vorticity mode at $\tilde{\alpha} = O(1)$. It is shown in the figure 6.

When the wall is cooling, C_1 increases rapidly causing destabilization of the flow.

5. The viscous counterpart of the inviscid vorticity mode as $M_\infty \to \infty$.

If we substitute the profiles of the undisturbed flow in the transition layer (3.7) and the function representations (4.5) into (3.1), it may be shown that the dissipative terms have the same order as the inviscid ones at the limit $\epsilon \to 0$, $(\tilde{\alpha}, c_t, \psi) = O(1)$ if the condition $R^* \equiv R \cos\psi\, C_1 \epsilon^\nu = O(1)$ is valid.

It is important to note here that we have found that the viscosity is not the first factor which should be taken into consideration as R diminishes. The first, for the no-interaction base flow, is a non-parallel effect induced by the transition-layer curvature. The curvature produces a centrifugal force field which influences the density fluctuations in the transition layer. The effect must be taken into account at $R^* = O(\epsilon^{-1/2k_2})$, and work on this is underway.

However, in this section the viscous counterpart of the inviscid vorticity mode is studied in the local-parallel approximation in the limit $R^* = O(1)$, $\epsilon \to 0$. There are many reasons to investigate this limit. The purpose of this study was to understand how viscosity changes the structure of the vorticity mode and to prove the existence of the mode in the presence of viscosity. Most of the numerical results at high Mach numbers were obtained using the local-parallel approximation. The 'high Mach number' is, in practice $M_\infty \approx 7$–20. We have made calculations up to $R^* = 10^6$, so at least a part of the results is correct at high Mach numbers. The results help to understand and explain some regularities found in previous numerical calculations.

Although the viscous vorticity-mode analysis in the local-parallel approximation is a correct asymptotic result, it does not hold for the boundary layer on the flat plate. The wall may be made slightly concave to eliminate the upper boundary edge curvature. Therefore the viscosity, curvature and interaction effects may be considered as different physical factors.

In the limit $R^* = O(1)$, $\epsilon \to 0$ we obtain the system of equations

$$\left.\begin{aligned} L_\phi f_t &= -P_t - \mathrm{i}\frac{u_t'}{T_t}\phi_t + \frac{1}{\tilde{\alpha}R^*}[\mathrm{i}(u_t' s_t)' + \tilde{\alpha}^2(\mu d_t + \mu'\phi_t)], \\ L_\phi \phi_t &= \mathrm{i}\frac{P_t'}{\tilde{\alpha}^2} - \frac{1}{\tilde{\alpha}R^*}[s_t u_t' + \mathrm{i}(\mu d_t' + \mu'\phi_t')], \\ L_\theta \theta_t &= \mathrm{i}\frac{T'}{T}\phi_t, \quad d_t = \frac{1}{\sigma\tilde{\alpha}R^*}\Delta^-(1)\,\mu\theta_t, \end{aligned}\right\} \tag{5.1}$$

i.e. the terms $(\tilde{U}-\tilde{c})\tilde{\pi}$ and $\tilde{D}$ are of $O(\cos^2\psi\, C_1^2 \epsilon^{2\nu-1})$ as $\epsilon \to 0$ and may be totally discarded in (3.1). The equation for h_t stands alone:

$$L_\phi h_t = -\mathrm{i}\frac{u_t'}{T_t}\phi_t + \frac{\mathrm{i}}{\tilde{\alpha}R^*}(s_t u_t')'.$$

The definition of $\Delta^-(\mu)$ corresponds to that previously stated:

$$L_\phi = \frac{c_t - u_t}{T_t} - \frac{1}{\mathrm{i}\tilde{\alpha}R^*}\Delta^-(\mu), \quad L_\theta = \frac{c_t - u_t}{T_t} - \frac{1}{\mathrm{i}\sigma\tilde{\alpha}R^*}\Delta^-(1)\,\mu,$$

$$d_t = \mathrm{i}f_t + \phi_t', \quad P_t = \pi_t + \tfrac{2}{3}(\mu - \lambda)\,(\tilde{\alpha}/R^*)\,d_t, \quad s_t = \frac{\mathrm{d}\mu}{\mathrm{d}T}\theta_t.$$

Equations (5.1) can be written as a system of six ordinary differential equations of the first order. The system has six linearly independent solutions. An analysis of the asymptotic structure of these six solutions as $y_t \to \pm\infty$ is given in Appendix B for

$y \to +\infty$ and $y \to -\infty$. It is shown that in both cases all the functions exponentially grow in three solutions and decay in the others. The requirement to discard the exponentially growing functions gives the boundary conditions for (5.1):

$$f_t, \phi_t, \theta_t \to 0 \quad \text{as} \quad y_t \to \pm\infty. \tag{5.2}$$

The functions P_t, f_t can be eliminated from (5.1). We have as a result a system of two equations:

$$\left.\begin{aligned} \Delta^-(L_\phi)\,\phi_t &= -\left(\frac{u_t'}{T_t}\phi_t\right)' + \frac{1}{\sigma\tilde{\alpha}R^*}(L_\phi\,\Delta^-(1)\,\mu\theta_t)' \\ &+ \frac{1}{\mathrm{i}\tilde{\alpha}R^*}[\tilde{\alpha}^2\mu''\phi_t + \mathrm{i}\Delta^+(1)\,(s_t u_t')' + \frac{\tilde{\alpha}}{\sigma R^*}\mu'\Delta^-(1)\,\mu\theta_t], \\ L_\theta\,\theta_t &= \mathrm{i}(T_t'/T_t)\,\phi_t. \end{aligned}\right\} \tag{5.3}$$

Evidently, (5.3) can be reduced to a single equation for θ_t. In this case, however, the equation contains both c_t and c_t^2, which is not suitable when the spectral methods are used in calculations.

Let us consider the fundamental features of the approximation constructed.

(i) The Mach number in the coordinate system moving with the wave speed

$$M = \tilde{M}^2\frac{(\tilde{U}-\tilde{c}_r)^2}{T} = \frac{2}{\gamma-1}\epsilon^{2\nu-1}C_1^2\cos^2\psi\frac{(u_t-c_{tr})^2}{T_t},$$

tends to zero in the transition layer. The function $\tilde{\pi}$ is discarded from the equation of state for disturbances $\tilde{\pi} = \tilde{r}T + \tilde{\theta}/T$, i.e. the total state equation is isobaric, the density perturbation is defined by the temperature disturbance. Because of this, the divergence of the velocity perturbation d_t is produced by a dissipative term (the last equation in (5.1)). Therefore, the divergence becomes zero (as for incompressible fluid) in the inviscid approach. If we remove the temperature fluctuations θ_t in the first equation of (5.3) and state $T = 1$, then the Orr–Sommerfeld equation will be obtained. Also, it should be noted that after eliminating P_t the problem does not contain the bulk viscosity λ at all.

(ii) When $\tilde{\alpha}$ is determined, the eigenvalues c_t and eigenfunctions ϕ_t, θ_t depend on three parameters only: σ, ω and $R^* = R\cos\psi C_1\epsilon^\nu$. The flow perturbations are localized in the thin region between the boundary layer and the free stream. The problem receives information from the boundary layer only indirectly from the constant C_1. From this we believe that the same problem statement holds for the vorticity mode in the hypersonic mixing layer, wake, jet – whenever there is an edge between the hypersonic free stream and the region of viscous gas with high enthalpy. This assumption was proved to a certain extent. The analyses of the inviscid instability in the hypersonic mixing layer (Balsa & Goldstein 1990) and in the boundary layer on a flat plate (Smith & Brown 1990) bring out the same problem statement.

The problem (5.3), (5.2) was investigated with the aid of a spectral method primarily developed for reliable and precise solving of the complete stability problem (3.1), (3.2), (3.3) at moderate and high Mach numbers. The method and its testing are described in detail in Grubin, Simakin & Trigub (1992). Some modifications are made to the method to solve (5.2), (5.3).

The variable η_t is used in numerical calculations. The region $\eta_a(\tilde{\alpha}, R^*) < \eta_t < \eta_b(\tilde{\alpha}, R^*)$ is transformed by

$$\eta_t = \eta_a + k\frac{1+z}{1-\beta z}, \quad \frac{dz}{d\eta_t} = \frac{(1-\beta z)^2}{k(1+\beta)}, \quad \beta = 1 - \frac{2k}{\eta_b - \eta_a}$$

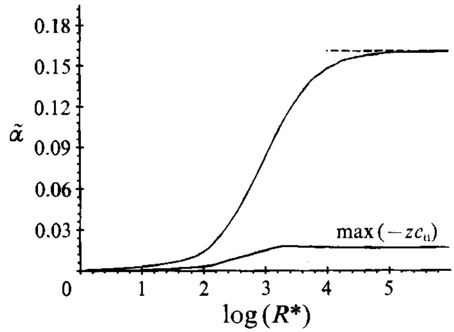

FIGURE 7. The universal upper branch of the neutral curves and the line of maximum amplification rate for the numerical solution of (5.2), (5.3). The inviscid vorticity-mode limit obtained from (4.1) is shown as a dashed line.

in the interval $-1 < z < 1$. The lower and upper points η_a, η_b are controlled to ensure strong decay of the functions ϕ, θ at the boundaries in accordance with the results of Appendix B. The functions are represented as

$$\phi = \sum_{n=0}^{Q} \hat{\phi}_n T_n(z),$$

and similarly for θ, where T_n are the Chebyshev polynomials. The equations (5.3) are approximated at the collocation points $z_i = \cos(\pi i/Q)$, $i = 1, \ldots Q-1$ and the boundary conditions (5.2) at points $z = -1, 1$. Parameter k may be used to move points in the (η_a, η_b) interval. The derivatives in the operators are obtained from

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}y_t} = \frac{\mathrm{d}z}{\mathrm{d}\eta_t}(z_i)\frac{1}{\sqrt{2T_t(z_i)}}\sum_{n=0}^{Q} DZ_{in}\hat{\phi}_n, \quad i = 1, \ldots Q-1,$$

where $DZ_{in} = T'_n(z_i)$. On using these procedures we obtain the eigenvalue problem for the system of linear equations

$$c_t A_{ij} X_j = B_{ij} X_j, \quad j = 0, \ldots 2Q+1, \quad i = 2, \ldots 2Q-1,$$

$$\sum_{j=0}^{Q} (-1)^j X_j = 0, \quad \sum_{j=Q+1}^{2Q+1} (-1)^j X_j = 0,$$

$$\sum_{j=0}^{Q} X_j = 0, \quad \sum_{j=Q+1}^{2Q+1} X_j = 0,$$

where $X_j = \phi_j$ and $X_{Q+j+1} = \theta_j$, $j = 0, \ldots Q$.

The spectral method was easy realized with the use of the spectral-method programs from the TURLEN Library. Different numbers of points Q were tested up to $Q = 100$. The final results are obtained at $Q = 60$.

It was proved that as $\tilde{\alpha} \to 0$, c_{tr}, $-c_{ti}$ tend to infinity but the maximum increment is at $\tilde{\alpha} = O(1)$. As $R^* \to 0$ the instability region rapidly becomes thin; however, the critical value R^* and the lower branch of the neutral curve were not found. The instability region apparently exists at the limit $R^* \to 0$. These results require analytical verification, i.e. the limit $\tilde{\alpha} = 0$, $R^* \to 0$ must be considered.

The upper branch of the neutral curve and the line of maximum rate of amplification are plotted in figure 7. The dashed line denotes the value of $\tilde{\alpha}$ for the neutral mode obtained from the solution of (4.1). The maximum rate of amplification

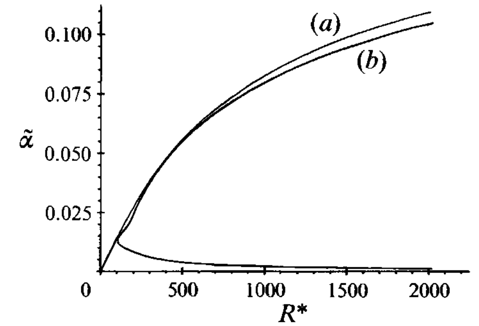

FIGURE 8. The universal upper branch of the neutral curves (*a*) and the neutral curve from the numerical solution of the complete system (3.1), (3.2), (3.3) at $M_\infty = 20$, $T_f = 1$, $\phi = 0$ (*b*).

increases with an increase of R^*. It is known from the calculation of the Lees–Lin equations for the boundary layer at $M \geqslant 4$ that at fixed values of Mach and Reynolds numbers the maximum amplification rate decreases with an increase of the wave angle ψ and increases with wall cooling. The theory produced explains these facts and states the similarity rules for the changes mentioned above.

Figure 8 shows a comparison of the universal neutral curve obtained from the Lees–Lin equations for $M_\infty = 20$, $\psi = 0$, $T_f = 1$. As already stated, the difference appears at small values of $\tilde{\alpha}$.

We wish to thank Dr I. N. Simakin for his consultation on the numerical methods. We are grateful to Dr S. Cowley for his comments and suggestions on this paper. We would also like to thank INTECO for support and assistance during the preparing of the paper for publishing.

Appendix A

Let $T_b = C_2 \xi^{-\lambda_2}(1+q(\xi)), f_b = \xi + C_1 \xi^{-\lambda_1}(1+r(\xi))$, then

$$\xi f_b'' = C_1 \lambda_1(\lambda_1+1)\xi^{-\lambda_1-1}(1+s(\xi)), \quad s = r - \frac{2}{\lambda_1+1}\xi r' + \frac{1}{\lambda_1(\lambda_1+1)}\xi^2 r''.$$

From (2.2) we have equations for the functions q, s:

$$\xi^2 q'' + \frac{1-3\omega}{1-\omega}\xi q' - 2\left(\frac{1+\omega}{1-\omega}\right) q = R_1, \quad \xi s' = R_2, \tag{A 1}$$

where

$$\left.\begin{aligned} R_1 = {} & (1-\omega)\frac{\xi q'^2}{1+q} + 2\frac{1+\omega}{(1-\omega)^2}\Big\{(1+q)^{1-\omega} - 1 - (1-\omega) q \\ & + ((1+q)^{1-\omega} - 1)\left(q - \frac{1-\omega}{2}\xi q'\right)\Big\} + 2\frac{1+\omega}{(1-\omega)^2}\Big\{C_1 \xi^{-\lambda_1 - 1}(1+r) \\ & \times \left(1 + q - \frac{1-\omega}{2}\xi q'\right)(1+q)^{1-\omega}\Big\} - \frac{4\sigma C_1^2 \lambda_1^2(\lambda_1+1)^2}{(1-\omega)\lambda_2 C_2}\xi^{-2\lambda_1-2+\lambda_2}(1+s)^2, \\ R_2 = {} & (1-\omega)\left(\frac{1+s}{1+q}\right)\xi q' - \lambda_1(1+s)[(1+C_1\xi^{-\lambda_1-1}(1+r))(1+q)^{1-\omega} - 1]. \end{aligned}\right\} \tag{A 2}$$

Series presentations of q, r, s are

$$q = \sum_{i=1}^{\infty} q_i \xi^{\nu_i}, \quad r = \sum_{i=1}^{\infty} r_i \xi^{\nu_i}, \quad s = \sum_{i=1}^{\infty} s_i \xi^{\nu_i},$$

$$s_i = \frac{(\nu_i - \lambda_1)(\nu_i - \lambda_1 - 1)}{\lambda_1(\lambda_1 + 1)} r_i.$$

The operator on the left-hand side of the first equation of (A 1) possesses the two-power eigensolution $q = \xi^{\beta}$ and $q = \xi^{\bar{\beta}}$:

$$\beta = -\frac{(2-\omega^2)^{\frac{1}{2}} - \omega}{1-\omega} < 0, \quad \bar{\beta} = \frac{(2-\omega^2)^{\frac{1}{2}} + \omega}{1-\omega} < 0,$$

of which only the first satisfies the condition $q \to 0$ as $\xi \to \infty$. Taking into account the existence of the eigensolution we can define the set of powers ν_i:

$$\begin{aligned}\{\nu_i\} = \{&(\beta, 2\beta, 3\beta, \ldots); (-\lambda_1 - 1, -\lambda_1 - 1 + \beta, -\lambda_1 - 1 + 2\beta, \ldots,);\\ &(-2\lambda_1 - 2 + \lambda_2, -2\lambda_1 - 2 + \lambda_2 + \beta, -2\lambda_1 - 2 + \lambda_2 + 2\beta, \ldots);\\ &(-2(\lambda_1 - 1), \ldots); \ldots\}.\end{aligned}$$

When ω and σ are determined this set must be put in order so that $\nu_{i+1} < \nu_i$. The right-hand sides of (A 2) are also presented as the series:

$$R_1 = \sum_{j=2}^{\infty} R_{1j} \xi^{\nu_j}, \quad R_2 = \sum_{j=1}^{\infty} R_{2j} \xi^{\nu_j}.$$

The coefficient $q_1 \equiv C_3$ is the free parameter in the expansions; the others are defined by the relations

$$q_j = R_{1j}/[(\nu_j - \beta)(\nu_j - \bar{\beta})], \quad j = 2, \ldots,$$
$$s_j = R_{2j}/\nu_j, \quad j = 1, \ldots,$$

so that for every j the right-hand sides depend on the $q_k, s_k, r_k, k < j$ only,

The expressions for the coefficients R_{1j}, R_{2j} were obtained by using symbolic computation on a computer.

Appendix B

We wish to investigate the behaviour of the solution of (5.1) as $y \to \pm\infty$.

When $y_t \to +\infty$ a particular case of the analysis arises, which was investigated by Mack (1969). The derivatives T_t', u_t' rapidly decay and at the leading-order as $y_t \to +\infty$ a system with constant coefficients is obtained. The solution can be written as

$$(f_t, \phi_t, \pi_t, \theta_t) = e^{\Omega y_t}(f, \phi, \pi, \theta), \tag{B 1}$$

where $f, \phi, \pi, \theta, \Omega$ are constants.

We can divide six linear independent solutions into three groups so that each of them contains exponentially growing and decaying functions as $y_t \to +\infty$:

(i) vorticity waves $\Omega^2 = \tilde{\alpha}^2 + i\tilde{\alpha}c_t R^*$:

$$\phi \neq 0, \quad f = +i\Omega\phi, \quad \theta = 0, \quad \pi = 0; \tag{B 2}$$

(ii) entropy waves $\Omega^2 = \tilde{\alpha}^2 + \mathrm{i}\sigma\tilde{\alpha}c_t R^*$:

$$\theta \neq 0, \quad f = \left(c_t + \mathrm{i}\frac{\Omega^2}{\sigma\tilde{\alpha}R^*}\right)\theta, \quad \phi = \frac{\Omega}{\sigma\tilde{\alpha}R^*}\theta, \quad \pi = -\mathrm{i}\tilde{\alpha}c_t\frac{1-\sigma}{\sigma}\frac{\theta}{R^*}; \tag{B 3}$$

(iii) potential waves $\Omega^2 = \tilde{\alpha}^2$:

$$\phi \neq 0, \quad f = \mathrm{i}\Omega\phi, \quad \theta = 0, \quad \pi = -\mathrm{i}\Omega c_t \phi. \tag{B 4}$$

As $y_t \to -\infty$ power-law growth of the profiles u_t, T_t is observed and coefficients in (5.1) are the power functions. In this case, finding the solution is more complex. Solutions of (5.1) should be sought, as $y_t \to -\infty$, in the form

$$(f_t, \phi_t, \pi_t, \theta_t) = (f, \phi, \pi, \theta)\exp\left(-\int_{y_t}^{y_{t0}} \Omega \, \mathrm{d}y\right), \tag{B 5}$$

where y_{t0} is some constant and the functions $f, \phi, \pi, \theta, \Omega$ are expanded in power series. We substitute (B 5) into (5.1) and use the representation (2.9) of the undisturbed profiles u_t, T_t as $y_t \to -\infty$. It is proved that the structures of the six linearly independent solutions as $y_t \to -\infty$ are similar to those obtained as $y_t \to +\infty$. There are three classes of solution, each containing exponentially growing and decaying functions. The same definitions as mentioned above are used:

(i) vorticity waves $\Omega^2 = \tilde{\alpha}^2 + \mathrm{i}(\tilde{\alpha}R^*/\mu T_t)(c_t - u_t)$:

$$\left.\begin{aligned}
\phi &= \frac{1}{\mu(\Omega T_t)^{\frac{1}{2}}(\Omega^2 - \tilde{\alpha}^2)}(1 + O(\xi^{1-k_1})), \quad \xi = -y_t, \\
f &= \mathrm{i}\Omega\phi(1 + O(\xi^{-1})), \\
\theta &= -\mathrm{i}\frac{\sigma}{1-\sigma}\frac{T'_t}{c_t - u_t}\phi(1 + O(\xi^{1-k_1})), \\
\pi &= \frac{\tilde{\alpha}\mu}{R^*}\left[2\frac{\mu'}{\mu} + \frac{T'_t}{T_t} - 2\frac{u'_t}{u_t - c_t}\right]\phi(1 + O(\xi^{1-k_1}));
\end{aligned}\right\} \tag{B 6}$$

(ii) entropy waves $\Omega^2 = \tilde{\alpha}^2 + (\mathrm{i}\sigma\tilde{\alpha}R^*/\mu T_t)(c_t - u_t)$:

$$\left.\begin{aligned}
\theta &= \frac{1}{\mu}\left(\frac{T_t}{\Omega}\right)^{\frac{1}{2}}(1 + O(\xi^{1-k_1})), \quad \phi = \frac{(\Omega T_t)^{\frac{1}{2}}}{\sigma\tilde{\alpha}R^*}(1 + O(\xi^{1-k_1})), \\
f &= \mathrm{i}\frac{\tilde{\alpha}}{\sigma R^*}\left(\frac{T_t}{\Omega}\right)^{\frac{1}{2}}(1 + O(\xi^{1-k_1})), \\
\pi &= -\mathrm{i}\frac{1-\sigma}{\sigma}\frac{\tilde{\alpha}}{R^*}\frac{c_t - u_t}{(\Omega T_t)^{\frac{1}{2}}}(1 + O(\xi^{1-k_1}));
\end{aligned}\right\} \tag{B 7}$$

(iii) potential waves $\Omega^2 = \tilde{\alpha}^2$:

$$\left.\begin{aligned}
&\phi = T_t^{\frac{1}{2}}(1 + O(\xi^{1-k_1})), \quad f = \mathrm{i}\Omega\phi(1 + O(\xi^{-1})), \\
&\pi = -\mathrm{i}\frac{\tilde{\alpha}^2(c_t - u_t)}{\Omega T_t}\phi(1 + O(\xi^{1-k_1})), \quad \theta = \mathrm{i}\frac{T'_t}{c_t - u_t}\phi(1 + O(\xi^{1-\kappa_1})).
\end{aligned}\right\} \tag{B 8}$$

The main approximation, obtained in this way, can be verified by substituting (B 5) in (5.1) and evaluating the remaining terms.

The higher-order terms ($c_t \ll u_t$) are preserved in the main approximation. This leads to a more general and useful form of representation. Expansions obtained can be considered as a generalization of presentation of the solution in the uniform free stream (B 1) in the case of external flow with power profiles u_t, T_t. The expansions (B 2), (B 3), (B 4) may be obtained from (B 6), (B 7), (B 8) as a particular case if we assume $u_t = 0$, $T_t = 1$.

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